Entanglement Entropy in a Non-conformal Field Theory

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[Ali-Akbari et al. (2022)]

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Outline

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Definition

If a pure bipartite quantum system is divided into two parts A and A^c, the total Hilbert space $\mathcal{H}_{tot}$ becomes factorized: ($\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$). We define the reduced density matrix ρ_A by $\rho_A = Tr_{A^c} \rho_{tot}$, where ρ_{tot} is the total density matrix $\rho_{tot} = |\psi\rangle\langle\psi|$ and $|\psi\rangle$ is a pure state in $\mathcal{H}_{tot}$. The entanglement entropy $S(\rho_A)$ for the subsystem A is defined by

$$S(\rho_A) = -Tr \rho_A \log \rho_A \quad \begin{cases} S = 0 & \Leftrightarrow \quad \rho_A \text{ is pure} \quad \Leftrightarrow \quad |\psi\rangle \text{ is separable state} \\ S > 0 & \Leftrightarrow \quad \rho_A \text{ is mixed} \quad \Leftrightarrow \quad |\psi\rangle \text{ is entangled state} \end{cases}$$

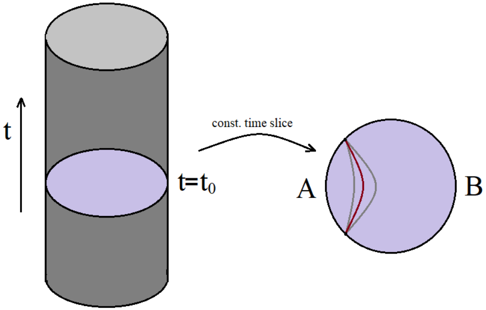
EE is a unique quantum measure for pure states.

[Horodecki et al. (2009)]

properties

- $S(\rho_A)$ is zero if and only if ρ_A represents a pure state.
- $S(\rho_A)$ is maximal and equal to $Log(N)$ for a maximally mixed state, N being the dimension of the Hilbert space.
- If ρ_{tot} is pure (i.e. $\rho_{tot} = |\psi\rangle\langle\psi|$), then $S(\rho_A) = S(\rho_B)$ and therefore, EE is not extensive.
- $S(\rho_A)$ is invariant under changes in the basis of ρ_A (i.e. $S(\rho_A) = (U\rho_A U^\dagger)$.
- For thermal state $S(\rho_A) = S_{th}$.
- EE includes UV divergences.

Holographic Entanglement Entropy



The holographic entanglement entropy (HEE) is given by [Ryu and Takayanagi (2006)]

$$S(\rho_A) = \frac{Area(\Gamma_A^{min})}{4G_N}. \tag{1}$$

Modified AdS and AdS black hole

We consider a holographic background in five dimensions, which is called MAdS and its black hole version MBH and dual to QCD-like theories at zero and finite temperature, respectively.

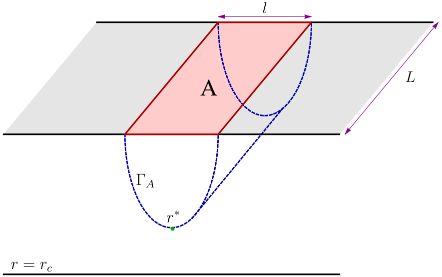
MAdS background described by the following metric [Andreev and Zakharov (2006)]

$$ds^2 = \frac{r^2}{R^2} g(r) \left(-dt^2 + d\vec{x}^2 + \frac{R^4}{r^4} dr^2 \right), \qquad g(r) = e^{\frac{r_c^2}{r^2}}, \qquad r_c = R^2 \sqrt{\frac{c}{2}}, \tag{2}$$

and the MBH background is described by [Andreev and Zakharov (2007)]

$$ds^2 = \frac{r^2}{R^2} g(r) \left(-f(r) dt^2 + d\vec{x}^2 + \frac{R^4}{r^4 f(r)} dr^2 \right), \qquad f(r) = 1 - \frac{r_H^4}{r^4}. \tag{3}$$

Configuration



We want to compute the HEE for a rectangular strip with width l and length $L(\to \infty)$, depicted in the above figure, specified by

$$-\frac{l}{2} \leq x(r) \leq \frac{l}{2}, \qquad -\frac{L}{2} \leq y \ \& \ z \leq \frac{L}{2}. \tag{4}$$

Zero temperature

Using the RT-prescription and MAdS background, the corresponding area surface is given by

$$\mathcal{A} = 2L^2 \int_{r^*}^{\infty} r e^{\frac{3}{2}\left(\frac{r_c}{r}\right)^2} \sqrt{1 + r^4 x'(r)^2} dr. \tag{5}$$

Since there is no explicit $x(z)$ dependence in (5), the corresponding Hamiltonian is constant and one can easily obtain

$$x'(r) = \pm \frac{r^{*3}}{r^5} e^{\frac{3}{2}\left(\left(\frac{r_c}{r^*}\right)^2 - \left(\frac{r_c}{r}\right)^2\right)} \left[1 - \left(\frac{r^*}{r}\right)^6 e^{3\left(\left(\frac{r_c}{r^*}\right)^2 - \left(\frac{r_c}{r}\right)^2\right)}\right]^{-\frac{1}{2}}. \tag{6}$$

By integrating the differential equation (6) and then plugging eq.(6) back into eq.(5), we find

$$l = \frac{2}{r^*} \int_0^1 u^3 e^{\frac{3}{2}\left(\frac{r_c}{r^*}\right)^2 (1-u^2)} \left[1 - e^{3\left(\frac{r_c}{r^*}\right)^2 (1-u^2)} u^6\right]^{-\frac{1}{2}} du, \tag{7}$$

$$\mathcal{A} = 2r^{*2} L^2 \int_{r^*\epsilon}^1 u^{-3} e^{\frac{3}{2}\left(\frac{r_c}{r^*}\right)^2 u^2} \left[1 - u^6 e^{3\left(\frac{r_c}{r^*}\right)^2 (1-u^2)}\right]^{-\frac{1}{2}} du, \tag{8}$$

where $u = \frac{r^*}{r}$ and ϵ is an ultraviolet cut off.

In order to calculate the above integrals, we use the generalized binomial expansion [Fischler and Kundu (2013)]

$$(1+x)^{-r} = \sum_{n=0}^{\infty} (-1)^n \binom{r+n-1}{n} x^n, \qquad |x| < 1. \tag{9}$$

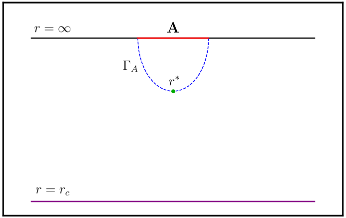
$$\tag{10}$$

Using eq.(9), we can write eqs. (7) and (8) as follows

$$l = \frac{2}{r^*} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)} \int_0^1 u^{6n+3} e^{3(n+\frac{1}{2})\left(\frac{r_c}{r^*}\right)^2 (1-u^2)} du, \tag{11}$$

$$\mathcal{A} = 2{r^*}^2 L^2 \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\Gamma(n+1)} \int_{r^*\epsilon}^1 u^{6n-3} e^{(3n+(\frac{3}{2}-3n)u^2)(\frac{r_c}{r^*})^2} du. \tag{12}$$

High energy limit



In the high energy limit, the energy scale corresponding to the subsystem A should be very larger than the energy scale Λ_c i.e. $\Lambda_c \ll \frac{1}{l}$. In terms of the bulk data, the high energy limit equals to $r_c \ll r^*$.

$$l = \frac{2}{r^*}\left[a_1 + a_2\left(\frac{r_c}{r^*}\right)^2 + a_3\left(\frac{r_c}{r^*}\right)^4\right], \qquad a_1, a_2, a_3 > 0, \tag{13}$$

$$S = \frac{1}{4G_N^{(5)}}\left(\frac{L}{\epsilon}\right)^2 - \frac{3}{8G_N^{(5)}}\Lambda_c^2 L^2 \log(\Lambda_c \epsilon) + S_{finite}(l, l\Lambda_c), \tag{14}$$

- The first term is a divergent term in the limit of $\epsilon \to 0$ which appears in the AdS background.
- From the second term we observe that there is a logarithmic divegence because of non-conformality.

$$\hat{S}_{finite}(l\Lambda_c) \equiv \frac{4G_N^{(5)} S_{finite}(l, l\Lambda_c)}{L^2\Lambda_c^2} = \frac{1}{(l\Lambda_c)^2}\Big[\kappa_1 + \left(\kappa_2 + \frac{3}{2}\log(l\Lambda_c)\right)(l\Lambda_c)^2 + \kappa_3 (l\Lambda_c)^4\Big], \quad \kappa_1 < 0, \kappa_2, \kappa_3 > 0, \ (15)$$

- The first term is the contribution of the AdS boundary corresponds to the entanglement entropy in conformal field theory. Obviously, this term is negative. [Velni et al. (2019)]
- The other two terms in eq.(15) are the non-conformal effects. In the second term the logarithmic term is the dominant term which is always negative in the high energy limit. This shows that the non-conformal effects decrease the HEE. [Rahimi et al. (2017)]

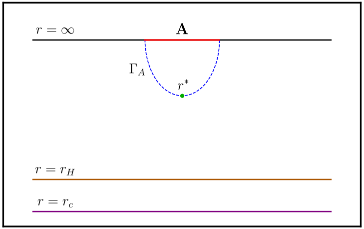
Finite temperature

By considering the MBH background and using the RT-prescription and then using the binomial expansion, the length of the subsystem and the area the RT-surface abtained as

$$l = \frac{2}{r^*}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})\Gamma(m+\frac{1}{2})}{\pi\Gamma(n+1)\Gamma(m+1)}\left(\frac{r_H}{r^*}\right)^{4m}\int_0^1 u^{6n+4m+3}e^{3(n+\frac{1}{2})(\frac{r_c}{r^*})^2(1-u^2)}du, \quad (16)$$

$$\mathcal{A} = 2{r^*}^2L^2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})\Gamma(m+\frac{1}{2})}{\pi\Gamma(n+1)\Gamma(m+1)}(\frac{r_H}{r^*})^{4m}\int_{r^*\epsilon}^1 u^{6n+4m-3}e^{(3n+(\frac{3}{2}-3n)u^2)(\frac{r_c}{r^*})^2}du. \quad (17)$$

High energy and low temperature



we focus on the limit of low temperature i.e $lT \ll 1$ at the high energy $l\Lambda_c \ll 1$. This regime can be interpreted in terms of bulk parameters as $r_H \ll r^*$ and $r_c \ll r^*$

$$l = \frac{2}{r^*}\left\{a_1 + b_1\left(\frac{r_H}{r^*}\right)^4 + \left[a_2 + b_2\left(\frac{r_H}{r^*}\right)^4\right]\left(\frac{r_c}{r^*}\right)^2 + \left[a_3 + b_3\left(\frac{r_H}{r^*}\right)^4\right]\left(\frac{r_c}{r^*}\right)^4\right\}, \qquad b_1, b_2, b_3 > 0, \tag{18}$$

$$\tilde{\mathcal{S}}_{finite}(l\Lambda_c, lT) \equiv \frac{4G_N^{(5)} S_{finite}(l, l\Lambda_c, lT)}{L^2 \Lambda_c T}$$
$$= \frac{1}{(l\Lambda_c)(lT)}\Big\{\kappa_1 + \bar{\kappa}_1 (lT)^4 + \left[\kappa_2 + \frac{3}{2}\log(l\Lambda_c) + \bar{\kappa}_2 (lT)^4\right](l\Lambda_c)^2$$
$$+ \left[\kappa_3 + \bar{\kappa}_3 (lT)^4\right](l\Lambda_c)^4\Big\}, \quad \bar{\kappa}_1, \kappa_2, \kappa_3 > 0, \kappa_1, \bar{\kappa}_2, \bar{\kappa}_3 < 0, \qquad (19)$$

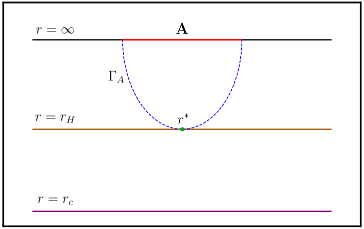
- The first two terms are the known results corresponding to the pure AdS and AdS black hole HEE, respectively. Since $\bar{\kappa}_1$ is a positive constant the second term is always positive and hence the thermal fluctuations increase $\tilde{\mathcal{S}}_{finite}$.
- The other two terms are the thermal and non-conformal corrections. In the third term the logarithmic term is dominant and always negative in the high energy limit. Therefore, the non-conformal effects decrease $\tilde{\mathcal{S}}_{finite}$.
- From eq.(19) we observe that if we fix lT ($l\Lambda_c$) and increases $l\Lambda_c$ (lT), then $\tilde{\mathcal{S}}_{finite}(l\Lambda_c, lT)$ will increase.
- In the limit of $lT \to 0$ we reach the MAdS results and in the limit of $l\Lambda_c \to 0$ we reach AdS black hole results.

We are interested in studying the HEE in the limit of $r_H \to r_c$ or equivalently $T \to \frac{\Lambda_c}{\sqrt{2}\pi}$, which we call it the transition limit. [Andreev and Zakharov (2007)]
We compare $\hat{\mathcal{S}}_{finite}(l\Lambda_c)$ at the zero temperature and $\tilde{\mathcal{S}}_{finite}(l\Lambda_c, lT)$ at low temperature in the transition limit

$$\left.\frac{\tilde{\mathcal{S}}_{finite}(l\Lambda_c, lT)}{\sqrt{2}\pi}\right|_{T\to\frac{\Lambda_c}{\sqrt{2}\pi}} - \hat{\mathcal{S}}_{finite}(l\Lambda_c) = \frac{\bar{\kappa}_1}{4\pi^4}(l\Lambda_c)^2 > 0. \tag{20}$$

- Near the transition point, the subsystem A and its complement $\bar{A}$ are less entangled at zero temperature.
- The EE describes the amount of information loss because of integrating out the subsystem $\bar{A}$. The higher the EE, the more information we lose.
- From the information point of view, we would like to define a favorable state such that the subsystems A and $\bar{A}$ are less entangled and hence near the transition point the state at zero temperature is the favorable one.

High energy and high temperature



$$\mathcal{S}^{(2)}_{HT}(l\Lambda_c, lT) \equiv \frac{4G_N^{(5)} S_{finite}(l, l\Lambda_c, lT)}{L^2 \Lambda_c T} = \frac{1}{(l\Lambda_c)(lT)} \Big\{ F_1 (lT)^2 + \pi^3 (lT)^3 + \left[F_2 + \frac{3}{2}\log(l\Lambda_c) \right] (l\Lambda_c)^2 \Big\}, \quad F_1 < 0, F_2 > 0. \quad (21)$$

- The first two terms correspond to HEE in the AdS black hole in the high temperature limit.
- The first term is area dependent, L^2 and the second term scales with the volume of the strip, lL^2 and hence the first term corresponds to the entanglement entropy between the strip region and its complement while the second term corresponds to the thermal entropy.
- The non-conformal effect appears in third term which is very small with respect to the first two terms. The logarithmic term in the braket is the dominant term and is always negative in the high energy limit. Hence the non-conformal effect decreases the HEE at high temperature in the high energy limit. [Rahimi et al. (2017)]
- In the limit of $l\Lambda_c \rightarrow 0$ we reach the results of the AdS black hole in the high temperature limit. [Velni et al. (2019)]

Thanks for your attention!